Superconducting hybrid extended objects

2001 J. Phys. A: Math. Gen. 34 3393

(http://iopscience.iop.org/0305-4470/34/16/307)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 3393–3402 www.iop.org/Journals/ja PII: S0305-4470(01)18366-6

Superconducting hybrid extended objects

Rubén Cordero and Alfonso Queijeiro

Departamento de Física, Escuela Superior de Física y Matemáticas del IPN Edificio 9, 07738, México DF, Mexico

Received 10 October 2000, in final form 5 February 2001

Abstract

The equations of motion for a relativistic extended object loaded with a superconducting edge are found in terms of geometrical quantities defined on the worldsheet. The results are applied to the study of a domain wall bounded by a superconducting string. Several cases for attaining equilibrium configurations are discussed.

PACS numbers: 9880C, 9880H, 1127

(Some figures in this article are in colour only in the electronic version; see www.iop.org)

1. Introduction

Superconducting cosmic strings have been widely studied due to its inherent interesting properties and cosmological consequences [1,2]. A dynamical study for these cosmic objects says that equilibrium configurations (called vortons or rings) may exist. These objects have been considered as candidates for explaining the existence of ultra-high-energy cosmic rays [3]. Besides, domain walls can exhibit a superconducting character too. Superconducting domain walls emerge from supersymmetry [4] and grand unified theories [5]. In addition, domain walls transform in superconducting membranes [6] in a similar way as in Witten's superconducting string [7]. In these objects, the presence of charge can lower the total energy density and they can be accommodated in standard cosmology [6]. On the other hand, a study of the mechanics of superconducting membranes from a Hamiltonian point of view has been carried out in [8].

Furthermore, hybrid objects (such as walls bounded by strings, and monopoles connected by strings) can be formed in an appropriate sequence of phase transitions. For example, the formation of domain walls bounded by strings is present in the Peccei–Quinn phase transition [9]. This kind of object has important physical consequences [10]. The dynamics of relativistic extended objects with a non-null edge (of the Dirac–Nambu–Goto type) have been studied in [11]. From a different point of view this problem is treated in [12]. It is important to study how the dynamical properties of superconducting string equilibrium configurations are modified if they are bounding the domain walls. The domain wall will change this equilibrium configurations in an unknown way.

In this paper we find the equations that describe the behaviour of a general kind of compound system (formed by a superconducting membrane and a superconducting edge) using the mathematical formalism developed in [13]. In particular we study the dynamical properties of a superconducting string bounding a domain wall and we show the existence of an equilibrium configuration.

The paper is organized as follows. In section 2 we consider the mathematical issues necessary to study these hybrid objects. In section 3 we obtain the fundamental equations of motion of the system. We focus in section 4 on a discussion about the mechanics of a superconducting string bounding a domain wall. The main conclusion of the work is presented in section 5. Finally, the variation of the action of this kind of composite system is developed explicitly in the appendix.

2. Mathematical background

The embedding of the worldsheet m describing the evolution of the $(p-1)$-dimensional extended object in an M background spacetime of dimension N and metric $g_{\mu\nu}$ can be defined by

$$x^\mu = X^\mu(\xi^a) \tag{1}$$

where x^μ are coordinates on M, ξ^a coordinates on m $(a, b = 0, \ldots, p-1)$ and X^μ the embedding functions. The tangent p vectors $e_a = X^\mu_{,a}\partial_\mu$ form a basis of tangent vectors to m at each point of m. The metric induced in the worldsheet from the background metric is given by

$$\gamma_{ab} = g(e_a, e_b) = X^\mu_{,a}X^\nu_{,b}g_{\mu\nu} \tag{2}$$

where a comma denotes simple partial differentiation with respect to the worldsheet coordinates ξ^a. The intrinsic geometry is determined by this metric. Tangential indices are manipulated with γ_{ab} and γ^{ab} in the usual way. The ith unit normal to the worldsheet, n^i $(i, j, \ldots = 1, N-p)$, is defined by

$$g(e_a, n^i) = 0 \qquad g(n^i, n^j) = \delta^{ij}. \tag{3}$$

Two of the most important quantities, determining the extrinsic geometry, are the extrinsic curvature K^i_{ab} of the worldsheet, defined by

$$K^i_{ab} = -g(D_a e_b, n^i) \tag{4}$$

where $D_a = e^\mu_a \nabla_\mu$, with ∇_μ the covariant derivative compatible with $g_{\mu\nu}$, and the extrinsic twist potential ω^{ij}_a, given by

$$\omega^{ij}_a = g(D_a n^i, n^j) = -\omega^{ji}_a \tag{5}$$

related to the covariance under normal frame rotations. We now consider some geometrical aspects of the worldsheet boundary ∂m. We treat ∂m as a timelike surface of dimension $p-1$, described by the embedding in the worldsheet m,

$$\xi^a = \chi^a(u^A) \tag{6}$$

where $A, B, \ldots = 0, 1, \ldots p-2$ and u^A are coordinates on ∂m. The $p-1$ tangent vectors to the boundary worldsheet are $\epsilon^a_A = \chi^a_{,A}$. Besides, in this case we have only one unit normal vector η^a to ∂m, defined by $\gamma_{ab}\eta^a\epsilon^b_A = 0$ and $\gamma_{ab}\eta^a\eta^b = 1$. The metric induced from m by the embedding χ^a is

$$h_{AB} = \gamma_{ab}\epsilon^a_A\epsilon^b_B. \tag{7}$$

In this case, the extrinsic geometry is determined only by the extrinsic curvature:

$$k_{AB} = -\gamma_{ab}\eta^a\nabla_A\epsilon^b_B \tag{8}$$

where $\nabla_A = \epsilon^a_A\nabla_a$, and ∇_a is the covariant derivative compatible with γ_{ab}. Note that in the case of a hypersurface embedding the extrinsic twist vanishes identically. For a complete study of this subject see [11].

3. Relativistic membranes with superconducting boundaries

The total action describing the dynamics of these hybrid objects in the presence of a background electromagnetic field is given as a sum of two parts corresponding to the membrane and its boundary

$$S \equiv S_0 + S_\mathrm{b} = \int_m \sqrt{-\gamma} L_0 + \int_{\partial m} \sqrt{-h} L_\mathrm{b} \tag{9}$$

where $L_0 = L_0(\gamma_{ab}, \bar{\varphi}_{,a}, A_a), L_\mathrm{b} = L_\mathrm{b}(h_{AB}, \varphi_{,A}, A_A)$, $\gamma \equiv \det\{\gamma_{ab}\}$, $h \equiv \det\{h_{AB}\}$, $A_a = \mathrm{e}_a^\mu A_\mu$ and $A_A = \epsilon_A^a A_a$ is the pullback of the external electromagnetic potential A_μ.

Note that, for the sake of simplicity, the worldsheet differential $d^p\xi$ has been absorbed into the integral sign. The Lagrangians L_0 and L_b depend on its own internal fields $\bar{\varphi}_{,a}, \varphi_{,A}$ and can depend on external fields, such as the electromagnetic external potential. In order to obtain the equations of motion we will perform a variation of the embedding of $X^\mu \rightarrow X^\mu + \delta X^\mu$. We can expand the displacement with respect to the spacetime basis $\{e_a, n^i\}$

$$\delta X = \Phi^a e_a + \Phi^i n_i. \tag{10}$$

Under this displacement the intrinsic metric change as [13]

$$\delta_X \gamma_{ab} = 2K_{ab}^i \Phi_i + \nabla_a \Phi_b + \nabla_b \Phi_a. \tag{11}$$

We are now in a position to perform the variation of the action. For the membrane the variation leads to

$$\delta_X S_0 = \int_m \sqrt{-\gamma} \left\{ \frac{1}{2} L_0 \gamma^{ab} \delta_X \gamma_{ab} + \frac{\delta L_0}{\delta \gamma_{ab}} \delta_X \gamma_{ab} + \frac{\delta L_0}{\delta A_a} \delta_X A_a \right\} \tag{12}$$

that can be cast in the form

$$\delta_X S_0 = \int_m \sqrt{-\gamma} \{ \tfrac{1}{2} T^{ab} \delta_X \gamma_{ab} + J^a \delta_X A_a \}. \tag{13}$$

The former expression specifies the electromagnetic current J_a and the electromagnetic energy stress tensor T^{ab}

$$J^a = \frac{\delta L_0}{\delta A_a} \qquad T^{ab} = 2\frac{\delta L_0}{\delta \gamma_{ab}} + L_0 \gamma^{ab} \tag{14}$$

where we consider that the dynamical equations for the internal fields are satisfied,

$$\nabla_a J^a = 0. \tag{15}$$

Using the results obtained in [13] for the variations of the geometrical quantities of the worldsheet, we find that the action takes the form (the explicit calculation of the following variations are given in the appendix)

$$\begin{aligned} \delta_X S_0 = \int_m \sqrt{-\gamma} \{ & (T^{ab} K_{ab}^i + F^i{}_a J^a) \Phi^i + (-\nabla_a T^{ab} + F^{ba} J_a) \Phi_b \\ & + \nabla_a (\Phi_j A^j J^a) + \nabla_a (T^{ab} \Phi_b + J^a A^b \Phi_b) \} \end{aligned} \tag{16}$$

where $F_a{}^i = e_a^\mu n^{i\nu} F_{\mu\nu}$, $F_{ab} = e_a^\mu e_b^\nu F_{\mu\nu}$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic tensor field. The tangential variation over the worldsheet yields as a result

$$\nabla_a T^{ab} = F^{ba} J_a. \tag{17}$$

This equation is satisfied identically as a consequence of the internal equations of motion (for the internal fields) of the membrane. The total divergences will be relevant for the motion of the boundary.

Performing now the variation of the action corresponding to the boundary, we obtain

$$\delta_X S_{\mathrm{b}} = \int_{\partial m} \left(\sqrt{-h}\delta_X L_{\mathrm{b}} + \delta_X \sqrt{-h} L_{\mathrm{b}}\right) \tag{18}$$

$$= \int_{\partial m} \sqrt{-h} \left\{ \frac{1}{2} L_{\mathrm{b}} h^{AB} \delta_X h_{AB} + \frac{\delta L_{\mathrm{b}}}{\delta h_{AB}} \delta_X h_{AB} + \frac{\delta L_{\mathrm{b}}}{\delta A_A} \delta_X A_A \right\} \tag{19}$$

$$= \int_{\partial m} \sqrt{-h} \{ \tfrac{1}{2} t^{AB} \delta_X h_{AB} + J^A \delta_X A_A \}. \tag{20}$$

In a similar way, the last expression specifies the momentum–energy tensor and the electromagnetic surface current over the boundary:

$$j^A = \frac{\delta L_{\mathrm{b}}}{\delta A_A} \qquad t^{AB} = 2\frac{\delta L_{\mathrm{b}}}{\delta h_{AB}} + L_{\mathrm{b}} h^{AB}. \tag{21}$$

This variation can be calculated if we use again the results in [13], and after tedious algebra (see appendix) we obtain the variation of the action corresponding to the boundary. The total variation is obtained when we combine the results (13) and (20):

$$\begin{aligned} \delta_X S = \int_m & \sqrt{-\gamma} \{ (T^{ab} K^i_{ab} + F^i{}_a J^a) \Phi_i \} \\ & + \int_{\partial m} \sqrt{-h} \{ (t^{AB} K^i_{AB} + j_b \mathcal{H}^{ab} F^i{}_a - A^i (\mathcal{H}^{ab} \nabla_a j_b - \eta^a J_a)) \Phi_i \\ & + (-t_{dc} k^{dca} + j_b \mathcal{H}^{bc} F_c{}^a - A^a (\mathcal{H}^{bc} \nabla_b j_c - \eta^b J_b) \\ & - (\mathcal{H}^{ac} \mathcal{H}^{bd} \nabla_b t_{dc} - T^{ab} \eta_b)) \Phi_a \} \end{aligned} \tag{22}$$

where $\mathcal{H}^{ab} = h^{AB} \chi^a_A \chi^b_B$. In order to obtain the equations of motion, we note that the coefficients of the deformation fields have to be zero. So, we have three equations of motions that describe the dynamics of the extended object with superconducting edges:

$$T^{ab} K^i_{ab} = F_a{}^i J^a \tag{23}$$

$$t^{AB} K^i_{AB} = F_a{}^i j_b \mathcal{H}^{ab} \tag{24}$$

$$t^{ab} k_{ab} = F_c{}^a \eta_a j_b \mathcal{H}^{bc} + T^{ab} \eta_a \eta_b \tag{25}$$

along with the conservation laws

$$\mathcal{H}^{ab} \nabla_a j_b = \eta^a J_a \tag{26}$$

$$\mathcal{H}^{ac} \mathcal{H}^{bd} \nabla_b t_{dc} = \mathcal{H}^a_c T^{bc} \eta_b + j_b \mathcal{H}^{bc} \mathcal{H}^a_d F^d_c. \tag{27}$$

The first equation describes the motion of the membrane. The second one is a kind of boundary condition of the first equation due to the boundary, and it is a restrictive condition over the motion of the membrane.

The equation (25) specifies the motion of the boundary, in which the effect of the membrane over the boundary has been taken into account through the orthogonal projection to the boundary of the stress energy tensor. The equation (27) is a consequence of the equations of motion of the internal fields (equations (15), (26)) and for this reason we do not need to resolve it, so the problem of the mechanics of this superconducting hybrid extended object is reduced to resolving the set of equations (15), (23)–(26).

4. Domain wall bounded by a superconducting string

Consider a system of a flat domain wall described by a Dirac–Nambu–Goto action bounded by a superconducting circular cosmic string in a four-dimensional background spacetime without external fields. In this case, from equations (23)–(26) we have to resolve the following equations (equation (15) is trivially satisfied since $J_a = 0$):

$$K^i = 0 \tag{28}$$

$$t^{AB} K^i_{AB} = 0 \tag{29}$$

$$k^{AB} t_{AB} = \mp\mu_0 \tag{30}$$

$$\mathcal{H}^{ab} \nabla_a j_b = 0 \tag{31}$$

where, in equation (30), the plus sign corresponds to the case in which the membrane has a hole whose boundary is the string itself, and the minus sign when the membrane has as boundary the superconducting string. Now, the dependence on $L_\mathrm{b} = L_\mathrm{b}(h_{AB}, \varphi_{,A}, A_A)$ is through $L_\mathrm{b} = L_\mathrm{b}(\omega)$, where $\omega = h^{AB}(\varphi_{,A} + A_A)(\varphi_{,B} + A_B)$.

Considering the case in which there are no external fields, $A_A = 0$, and the metric is flat

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2) \tag{32}$$

we obtain the flat domain wall if we put $\theta = \pi/2$

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\, \mathrm{d}\phi^2 \tag{33}$$

and the superconducting circular string through the embedding

$$t = E\tau \qquad r = r(\tau) \qquad \phi = \sigma \tag{34}$$

where E is a constant with dimension of length^{-1} in natural units; τ and σ are the string worldsheet coordinates. For the flat domain wall the extrinsic curvature identically vanishes, $K_{ab} = 0$, and equations (28) and (29) are trivially satisfied; it remains to resolve equations (30) and (31).

Now, we consider the situation in which the superconducting string has charge and current

$$\varphi = \varphi(\tau, \sigma) = \varphi^1(\tau) + N\sigma \tag{35}$$

where N is the winding number.

The equation for the internal field φ is equivalent to current conservation (31),

$$\dot{\varphi} = \frac{\Omega\sqrt{E^2 - \dot{r}^2}}{2r(\mathrm{d}L_\mathrm{b}/\mathrm{d}\omega)} \tag{36}$$

where Ω is a dimensionless integration constant. Then,

$$\omega = -\frac{\Omega^2}{4r^2(\mathrm{d}L_\mathrm{b}/\mathrm{d}\omega)^2} + \frac{N^2}{r^2}. \tag{37}$$

In order to calculate the extrinsic curvature we need to know the vectors tangent and normal to the worldsheet.

For our system these vectors are

$$\epsilon^a_0 = (E, \dot{r}, 0) \tag{38}$$

$$\epsilon^a_1 = (0, 0, 1) \tag{39}$$

$$\eta = 1/\sqrt{E^2 - \dot{r}^2}(\dot{r}, E, 0). \tag{40}$$

The non-vanishing components of the extrinsic curvature $k_{AB} = -g(\nabla_A \epsilon_B, \eta)$ are

$$k^{00} = \frac{\ddot{r}E}{(E^2 - \dot{r}^2)^{5/2}} \tag{41}$$

$$k^{11} = -\frac{E}{r^3(E^2 - \dot{r}^2)^{1/2}} \tag{42}$$

in such a way that the only equation of motion that we need to resolve is

$$L_{\mathrm{b}}k - 2\frac{\mathrm{d}L_{\mathrm{b}}}{\mathrm{d}\omega}k^{00}\varphi_{,0}\varphi_{,0} - 2\frac{\mathrm{d}L_{\mathrm{b}}}{\mathrm{d}\omega}k^{11}\varphi_{,1}\varphi_{,1} = \mp\mu_0. \tag{43}$$

Explicitly,

$$L_{\mathrm{b}}\left(-\frac{\ddot{r}E}{(E^2-\dot{r}^2)^{3/2}} - \frac{E}{r(E^2-\dot{r}^2)^{1/2}}\right) - \frac{\ddot{r}E\Omega^2}{(E^2-\dot{r}^2)^{3/2}2r^2(\mathrm{d}L_{\mathrm{b}}/\mathrm{d}\omega)}$$

$$+2\frac{\mathrm{d}L_{\mathrm{b}}}{\mathrm{d}\omega}\frac{EN^2}{r^3(E^2-\dot{r}^2)^{1/2}} = \mp\mu_0 \tag{44}$$

or, in an equivalent form,

$$\frac{\ddot{r}}{(E^2-\dot{r}^2)^{3/2}}\left(L_{\mathrm{b}} + \frac{\Omega^2}{2r^2\,\mathrm{d}L_{\mathrm{b}}/\mathrm{d}\omega}\right) + \frac{L_{\mathrm{b}}}{r(E^2-\dot{r}^2)^{1/2}} - 2\frac{\mathrm{d}L_{\mathrm{b}}}{\mathrm{d}\omega}\frac{N^2}{r^3(E^2-\dot{r}^2)^{1/2}} = \pm\frac{\mu_0}{E}. \tag{45}$$

Finally, the last equation can be written as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{r(L_{\mathrm{b}} + \frac{\Omega^2}{2r^2\,\mathrm{d}L_{\mathrm{b}}/\mathrm{d}\omega})}{(E^2-\dot{r}^2)^{1/2}}\right) = \pm\dot{r}r\frac{\mu_0}{E} = \pm\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mu_0 r^2}{2E}\right) \tag{46}$$

and we obtain a first integral

$$\dot{r}^2 = E^2 - \frac{r^2\left(L_{\mathrm{b}} + \frac{\Omega^2}{2r^2\,\mathrm{d}L_{\mathrm{b}}/\mathrm{d}\omega}\right)^2}{\left(\frac{E_T}{E} \mp \frac{\mu_0 r^2}{2E}\right)^2} \tag{47}$$

where E_T is a constant related to the total energy. When $\mu_0 = 0$ we recover the equation of motion for superconducting circular strings [14, 15].

The choice of the sign depends on whether the membrane has holes whose boundary is the string itself or the membrane has as boundary the string. So, we have shown that the solution to this problem reduces to a one-dimensional effective potential for the radius of the string.

One of the better studied models is Witten's one [7] described by the following Lagrangian: $L_{\mathrm{b}} = -(\mu_{\mathrm{b}} + \omega/2)$. The effective potential in this model is $V^* = \frac{E^2(N^2+\Omega^2)\mu_{\mathrm{b}}}{2E_T^2}V(a, x)$, which is a function of the dimensionless parameter $x = (\frac{2\mu_{\mathrm{b}}}{N^2+\Omega^2})^{1/2}r$ and $a = \frac{(N^2+\Omega^2)\mu_0}{2E_T\mu_{\mathrm{b}}}$. Since both x and a depend on the wall and string properties, we have plotted $V(a, x)$ in figure 1.

We see that the potential exhibits a minimum; i.e. there exists an equilibrium configuration for this hybrid object. We also note that for increasing values of the parameter a (meaning greater values for μ_0, keeping all other parameters fixed) the equilibrium configuration occurs for decreasing string radius, due to the tension on the wall, which is physically expected. However, Coulomb and centrifugal effects present in the string compensate the tension on the wall (and on the string), avoiding collapsing to a charge point.

On the other hand, if we consider the case in which the superconducting string is the boundary of a hole in the domain wall, we have to choose the plus sign in the expression for the one-dimensional potential. This potential is plotted in figure 2.

In this case we see that the potential lacks a minimum and therefore it does not possess an equilibrium configuration. According to figure 2, the hole grows indefinitely. This is so, because of the tension of the wall and the Coulomb and centrifugal repulsion overcome the tension of the string. Although the electromagnetic and centrifugal forces of the string decrease as the radius of the hole grows, the tension on the wall is responsible for this behaviour.

We have also studied Nielsen's model [16] and found that the effective potential behaves qualitatively as in Witten's case.

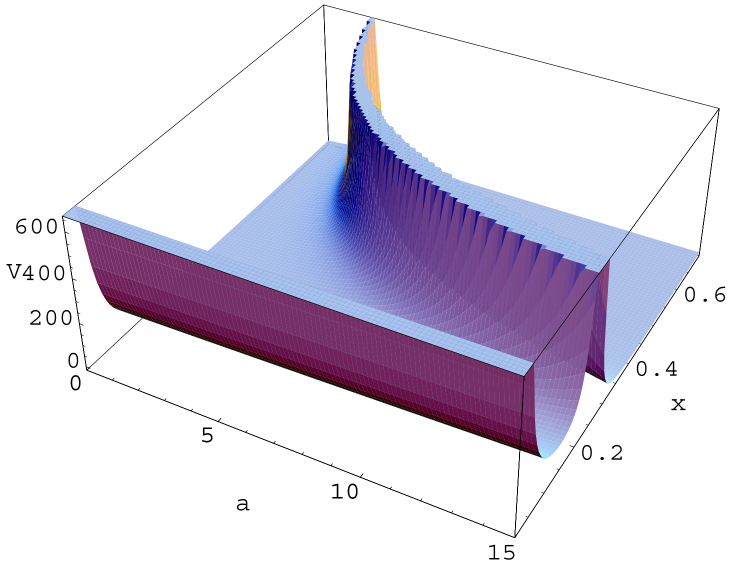

Figure 1. Effective potential for a superconducting circular string bounding a flat domain wall.

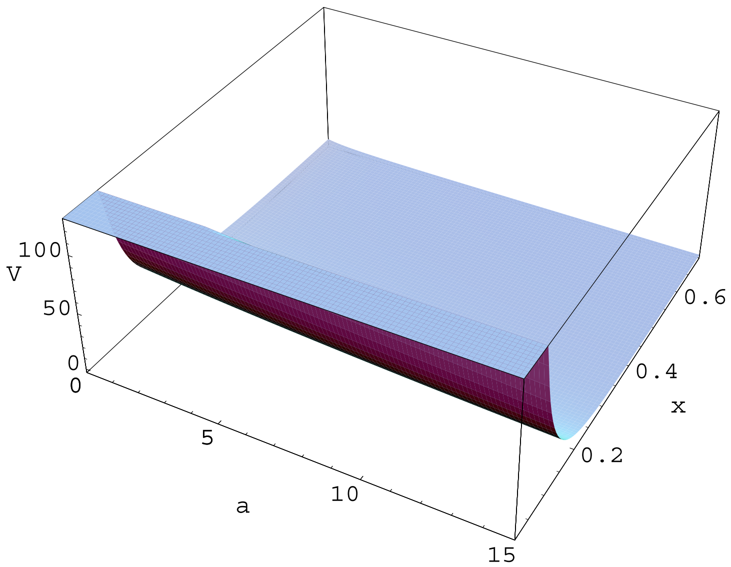

Figure 2. Effective potential for a superconducting circular string as the boundary of a hole in a flat domain wall.

5. Conclusion

In this paper we have obtained from a geometrical point of view the equations of motion for a hybrid superconducting extended object. Specifically, we have tackled a composite system of a flat domain wall bounded by a superconducting circular string. For this particular case, our problem is expressed by means of a one-dimensional effective potential and we have shown that there exists an equilibrium configuration. Such a solution might be considered as a candidate to explain the ultra-high-energy cosmic rays [10]; this issue is under current investigation.

Acknowledgments

We gratefully acknowledge support from CONACYT, SNI (México) and COFAA-IPN. RC is indebted to R Capovilla for many valuable suggestions and thanks X Martin and E Rojas for useful comments.

Appendix

In this appendix the computation of the total variation of the composite action

$$S = S_0 + S_{\mathrm{b}} = \int_m \sqrt{-\gamma} L_0 + \int_{\partial m} \sqrt{-h} L_{\mathrm{b}} \tag{48}$$

is developed explicitly.

We can perform the variation of the action in terms of the spacetime basis $\{e_a, n^i\}$. This variation can be expressed as an expansion of the tangential and normal deformation components

$$\delta X = \Phi^a e_a + \Phi^i n_i. \tag{49}$$

In order to obtain the equations of motion of this composite system, we will use the following results for the variation of the important quantities [13]:

$$\delta_X e_a^\nu = \{\nabla_a \Phi^c + K_a^{ci} \Phi_i\} e_c^\nu + \{\tilde{\nabla}_a \Phi_j - K_{acj} \Phi^c\} n^{\nu j} \tag{50}$$

$$\delta_X \gamma_{ab} = 2K_{ab}^i \Phi_i + \nabla_a \Phi_b + \nabla_b \Phi_a \tag{51}$$

$$\delta_X \gamma^{ab} = -\gamma^{ac} \gamma^{bd} \delta_X \gamma_{cd} \tag{52}$$

$$\delta_X \gamma = \gamma \gamma^{ab} \delta_X \gamma_{ab} \tag{53}$$

where $\tilde{\nabla}_a$ is the covariant derivative associated with the extrinsic twist potential ω_a^{ij}. Additionally, the variation of the external vector potential can be written in terms of the tangential and normal deformation as

$$\delta_X A_\nu = (\Phi_i n^{i\mu} + \Phi^a e_a^\mu) \nabla_\mu A_\nu. \tag{54}$$

Applying the former results to our variation procedure we obtain for the membrane

$$\begin{aligned} \delta_X S_0 = \int_m \sqrt{-\gamma} \{ & T^{ab}(K_{ab}^i \Phi_i + \nabla_a \Phi_b) + J^a A_c \nabla_a \Phi^c + K_a^{ci} J^a A_c \Phi_i \\ & + J^a A^j \tilde{\nabla}_a \Phi_j - J^a K_{acj} \Phi^c A^j + J^a e_a^\mu \Phi_i n^{i\nu} \nabla_\nu A_\mu + J^a e_a^\mu \Phi^b e_b^\nu \nabla_\nu A_\mu \}. \end{aligned} \tag{55}$$

Inserting the following expression in the last equation

$$\tilde{\nabla}_a A^i = K_{ab}^i A^b + e_a^\rho n^{i\mu} \nabla_\rho A_\mu \tag{56}$$

we find that the variation takes the form

$$\begin{aligned} \delta_X S_0 = \int_m \sqrt{-\gamma} \{ & T^{ab} K_{ab}^i \Phi_i + \nabla_a (\Phi_j A^j J^a) - J^a e_a^\rho \Phi_i n^{i\mu} \nabla_\rho A_\mu \\ & + J^a \Phi_i e_a^\mu n^{\nu i} \nabla_\nu A_\mu + T^{ab} \nabla_a \Phi_b + \nabla_a (\Phi_b J^a A^b) \\ & - J^a \Phi^b \nabla_a A_b - J^a K_{acj} A^j \Phi^c + J^a e_a^\mu \Phi^b e_b^\nu \nabla_\nu A_\mu \}. \end{aligned} \tag{57}$$

Working out the last expression we arrive at

$$\begin{aligned} \delta_X S_0 = \int_m \sqrt{-\gamma} \{ & (T^{ab} K_{ab}^i + F^i{}_a J^a) \Phi_i + (-\nabla_a T^{ab} + F^{ba} J_a) \Phi_b \\ & + \nabla_a (\Phi_j A^j J^a) + \nabla_a (T^{ab} \Phi_b + J^a A^b \Phi_b) \}. \end{aligned} \tag{58}$$

The vanishing of the tangential part gives, as a result, the relation

$$\nabla_a T^{ab} = F^{ba} J_a \tag{59}$$

which is identically fulfilled as a consequence of the equation of motion for the internal field. The variation corresponding to the boundary is connected to the variation of the membrane itself via

$$\delta_X h_{AB} = \chi^a_A \chi^b_B \delta_X \gamma_{ab}. \tag{60}$$

We need to define the following relation, which is a kind of projection over the boundary:

$$\mathcal{H}^{ab} = h^{AB} \chi^a_A \chi^b_B. \tag{61}$$

Performing the variation corresponding to the boundary, we obtain

$$\begin{aligned}\delta_X S_{\mathrm{b}} = \int_{\partial m} \sqrt{-h}\{& t^{AB} \chi^a_A \chi^b_B (K^i_{ab}\Phi_i + \nabla_a \Phi_b) + \mathcal{H}^{ab} j_b A_c \nabla_a \Phi^c \\ &+ K^{ci}_a \mathcal{H}^{ab} j_b A_c \Phi_i + \mathcal{H}^{ab} j_b A^j \nabla_a \Phi_j - j_b \mathcal{H}^{ab} K_{acj} \Phi^c A^j \\ &+ \mathcal{H}^{ab} j_b e^\mu_a \Phi_i n^{i\nu} \nabla_\nu A_\mu + \mathcal{H}^{ab} j_b e^\mu_a \Phi^c e^\nu_c \nabla_\nu A_\mu\}. \end{aligned}\tag{62}$$

Taking into account the following relations:

$$\begin{aligned} t^{AB}\chi^a_A \chi^b_B \nabla_a \Phi_b &= \mathcal{H}^{ac}\mathcal{H}^{bd} t_{cd} \nabla_a \Phi_b \\ &= \mathcal{D}_A(\epsilon^{dA}\mathcal{H}^{ac} t_{cd}\Phi_a) + k\eta^d \mathcal{H}^{ac} t_{dc}\Phi_a - \Phi_b \mathcal{H}^{bd}\nabla_a(\mathcal{H}^{ac} t_{dc}) \\ &= \mathcal{D}_A(\epsilon^{dA}\mathcal{H}^{ac} t_{cd}\Phi_a) + k\eta^d \mathcal{H}^{ac} t_{dc}\Phi_a - \Phi_a t_{dc} k^{dac} \\ &\quad - \Phi_a t_{dc} k^{dca} - \Phi_b \mathcal{H}^{bd}\mathcal{H}^{ac}\nabla_a t_{dc} \end{aligned}\tag{63}$$

and

$$\begin{aligned} \mathcal{H}^{ab} j_b A_c \nabla_a \Phi^c &= \mathcal{H}^{ab}\nabla_a(\Phi^c j_b A_c) - A_c \Phi^c \mathcal{H}^{ab}\nabla_a j_b - \mathcal{H}^{ab}\Phi^c j_b \nabla_a A_c \\ &= \mathcal{D}_A(\epsilon^{bA} j_b \Phi^c A_c) + k\eta^a j_a \Phi^b A_b - A_c \Phi^c \mathcal{H}^{ab}\nabla_a j_b - \mathcal{H}^{ab}\Phi^c j_b \nabla_a A_c \\ \mathcal{H}^{ab} j_b A^i \tilde{\nabla}_a \Phi_i &= \mathcal{H}^{ab}\nabla_a(\Phi_i j_b A^i) - A^i \Phi_i \mathcal{H}^{ab}\nabla_a j_b - \mathcal{H}^{ab}\Phi_i j_b \tilde{\nabla}_a A^i \end{aligned}\tag{64}$$

with

$$\mathcal{H}^{ab}\nabla_a(\Phi_i j_b A^i) = \mathcal{D}_A(\Phi_i j_b A^i \epsilon^{bA}) + k\eta^a A^i \Phi_i j_a$$

where $k_{AB} = k^c_{ab}\epsilon^a_A \epsilon^b_B \eta_c$, $k = h^{AB} k_{AB}$ and $\mathcal{D}_A$ is the covariant derivative compatible with h_{AB}, we can write the variation of the boundary action as

$$\begin{aligned}\delta_X S_{\mathrm{b}} = \int_{\partial m} \sqrt{-h}\{& t^{AB} k^i_{AB}\Phi_i + \mathcal{D}_A(\epsilon^{dA}\mathcal{H}^{ac} t_{cd}\Phi_a) + k\eta^d \mathcal{H}^{ac} t_{dc}\Phi_a \\ &- \Phi_a t_{dc} k^{dac} - \Phi_a t_{dc} k^{dca} - \Phi_a \mathcal{H}^{bd}\mathcal{H}^{ac}\nabla_b t_{dc} + \mathcal{D}_A(\epsilon^{bA} j_b \Phi^c A_c) \\ &+ k\eta^a j_a \Phi^b A_b - A_c \Phi^c \mathcal{H}^{ab}\nabla_a j_b - \mathcal{H}^{ab}\Phi^c j_b \nabla_a A_c + j_b \mathcal{H}^{ab} K^{ci}_a A_c \Phi_i \\ &+ \mathcal{D}_A(\Phi_i j_b A^i \epsilon^{bA}) + k\eta^a j_a A^i \Phi_i - A^i \Phi_i \mathcal{H}^{ab}\nabla_a j_b - \mathcal{H}^{ab}\Phi_i j_b n^{i\mu} D_a A_\mu \\ &- \mathcal{H}^{ab}\Phi_i j_b K^i_{ac} A^c - j_b \mathcal{H}^{ab} A^i K_{aci}\Phi^c + \mathcal{H}^{ab} j_b e^\mu_a \Phi_i n^{i\nu}\nabla_\nu A_\mu \\ &+ \mathcal{H}^{ab} j_b e^\mu_a \Phi^c e^\nu_c \nabla_\nu A_\mu\}. \end{aligned}\tag{65}$$

Considering that the total divergence terms do not contribute to the equations of motion, that

$$\begin{aligned} \eta_a j^a &= 0 \\ \eta^d t_{dc} &= 0 \\ t_{dc} k^{dac} &= 0 \end{aligned}\tag{66}$$

expressing that the current j is on the boundary only, t_{ab} is the boundary stress energy tensor and the relation

$$e^\mu_a D_c A_\mu = \nabla_c A_a + K^i_{ca} A_i \tag{67}$$

we have that the total variation simplifies to

$$
\begin{aligned}
\delta_X S = \int_m & \sqrt{-\gamma}\{(T^{ab}K^i_{ab} + F^i{}_a J^a)\Phi_i\} \\
& + \int_{\partial m} \sqrt{-h}\{(t^{AB}K^i_{AB} + j_b \mathcal{H}^{ab} F^i{}_a - A^i(\mathcal{H}^{ab}\nabla_a j_b - \eta^a J_a))\Phi_i \\
& +(-t_{dc}k^{dca} + j_b \mathcal{H}^{bc} F_c{}^a - A^a(\mathcal{H}^{bc}\nabla_b j_c - \eta^b J_b) \\
& -(\mathcal{H}^{ac}\mathcal{H}^{bd}\nabla_b t_{dc} - T^{ab}\eta_b))\Phi_a\}.
\end{aligned} \tag{68}
$$

Finally, the equations of motion describing the dynamics of the extended objects with charge boundaries are

$$T^{ab}K^i_{ab} = F_a{}^i J^a \tag{69}$$

$$t^{AB}K^i_{AB} = F_a{}^i j_b \mathcal{H}^{ab} \tag{70}$$

$$t^{ab}k_{ab} = F_c{}^a \eta_a j_b \mathcal{H}^{bc} + T^{ab}\eta_a\eta_b \tag{71}$$

along with the conservation laws

$$\mathcal{H}^{ab}\nabla_a j_b = \eta^a J_a \tag{72}$$

$$\mathcal{H}^{ac}\mathcal{H}^{bd}\nabla_b t_{dc} = \mathcal{H}^a_c T^{bc}\eta_b + j_b \mathcal{H}^{bc}\mathcal{H}^a_d F_c{}^d. \tag{73}$$

However, the last relation is a consequence of the equations that satisfy the internal fields over the boundary, i.e. it is a consequence of the equations (72) and (15); so we do not need this relation to resolve our problem.

References

[1] Vilenkin A and Shellard E P S 1994 *Cosmic Strings and Other Topological Defects* (Cambridge: Cambridge University Press)
[2] Vilenkin A 1985 *Phys. Rep.* **121** 263
[3] Bonazzola S and Peter P 1997 *Astropart. Phys.* **7** 161
[4] Morris J 1995 *Phys. Rev.* D **52** 1096
[5] Kibble T, Lazarides G and Shafi Q 1982 *Phys. Rev.* D **26** 435
[6] Peter P 1996 *J. Phys. A: Math. Gen.* **29** 5125
[7] Witten E 1985 *Nucl. Phys.* B **57** 256
[8] Cordero R and Rojas E 1999 *Phys. Lett.* B **470** 45
[9] Peccei R D and Quinn H R 1977 *Phys. Rev. Lett.* **38** 1440
[10] Martin X and Vilenkin A 1997 *Phys. Rev. Lett.* **77** 2879
Martin X and Vilenkin A 1997 *Phys. Rev.* D **55** 6054
Berezinsky V, Martin X and Vilenkin A 1997 *Phys. Rev.* D **56** 2024
[11] Capovilla R and Guven J 1997 *Phys. Rev.* D **55** 2308
Capovilla R and Guven J 1998 *Phys. Rev.* D **57** 5158
[12] Carter B 1995 *Formations and Interections of Topological Defects (NATO)* ed R Brandenberger and A C Davis (New York: Plenum)
[13] Capovilla R and Guven J 1995 *Class. Quantum Grav.* **12** L107
[14] Carter B, Peter P and Gangui A 1997 *Phys. Rev.* D **55** 4647
[15] Larsen A L and Axenides M 1997 *Class. Quantum Grav.* **14** 443
[16] Nielsen N K 1980 *Nucl. Phys.* B **167** 249